Worksheet: Defining one relation from another

We work in ordinary first-order logic with identity, with:

- monadic (unary) predicate symbols P, Q,
- a binary predicate symbol R,
- and, later, a binary predicate symbol S.

Throughout, x, y, z range over objects in the domain.

This worksheet explores what we can *prove* about a binary relation when it is defined in terms of simpler predicates.

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1. Defining a Product Relation

Suppose we *define* a binary relation R by:

$$Rxy \;\leftrightarrow\; Px \;\wedge\; Qy.$$

Intuitively, R relates exactly those pairs (x, y) such that x has property P and y has property Q.

Tasks

(1.1) Show that for all x and y:

$$Rxy \to Px \qquad \text{and} \qquad Rxy \to Qy.$$

(A very easy warm-up.)

(1.2) Define two unary predicates *from* R:

$$P_R(x) \;\equiv\; \exists y\, Rxy \qquad \text{and} \qquad Q_R(y) \;\equiv\; \exists x\, Rxy.$$

Show that in any structure satisfying the definition $Rxy \leftrightarrow Px \wedge Qy$, we have:

$$\forall x\big(P_R(x) \leftrightarrow Px\big) \qquad \text{and} \qquad \forall y\big(Q_R(y) \leftrightarrow Qy\big).$$

(1.3) **Rectangle Law.** Show that R satisfies the following "rectangle" property:

$$\forall x \forall x' \forall y \forall y' \big((Rxy \wedge Rx'y') \to (Rxy' \wedge Rx'y)\big).$$

In words: whenever (x, y) and (x', y') are in R, then the "crossed" pairs (x, y') and (x', y) are also in R.

(1.4) Explain why the Rectangle Law expresses the idea that the extension of R in the domain is a *full rectangular block* between the P-objects and the Q-objects. (It has no "holes" inside that block.)

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2. Defining an Equivalence from a Monadic Predicate

Now suppose we define a new binary relation R from a single unary predicate P:

$$Rxy \;\leftrightarrow\; \big(Px \leftrightarrow Py\big).$$

Intuitively: x is R-related to y iff x and y either both have property P, or both lack property P.

Tasks

(2.1) Prove that R is **reflexive**:

$$\forall x\, Rxx.$$

(Hint: what is $Px \leftrightarrow Px$?)

(2.2) Prove that R is **symmetric**:

$$\forall x \forall y\, (Rxy \to Ryx).$$

(Hint: use the symmetry of the biconditional: $Px \leftrightarrow Py$ iff $Py \leftrightarrow Px$.)

(2.3) Prove that R is **transitive**:

$$\forall x \forall y \forall z \big((Rxy \wedge Ryz) \to Rxz\big).$$

(Hint: if Px and Py have the same truth-value, and Py and $P(z)$ have the same truth-value, then Px and Pz have the same truth-value.)

(2.4) Conclude that R is an **equivalence relation**.

(2.5) Describe informally what the R-equivalence classes look like. (How many equivalence classes are there? Which objects are in each class?)

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3. Defining a Relation from Another

Now start with an arbitrary binary relation R. Define a new binary relation S by:

$$Sxy \leftrightarrow \forall z\big(Rxz \to Ryz\big).$$

Intuitively:

- Sxy means: *every* R-successor of x is also an R-successor of y.
- So y has at least all the R-successors that x has (perhaps more).

Tasks

(3.1) Prove that S is **reflexive**:

$$\forall x\, Sxx.$$

(Hint: for any x and z, $Rxz \to Rxz$ is always true.)

(3.2) Prove that S is **transitive**:

$$\forall x \forall y \forall w\big((Sxy \land Syw) \to Sxw\big).$$

(Hint: unpack the definition: if every R-successor of x is an R-successor of y, and every R-successor of y is an R-successor of w, then what can you say about the R-successors of x and w?)

(3.3) Is S necessarily **symmetric**? Either:

- give a proof that $\forall x \forall y(Sxy \to Syx)$ is valid, or
- give a countermodel (a structure and an interpretation of R) where Sxy holds but Syx fails for some x, y.

(3.4) Based on your answers above, what kind of relational structure is S? (For example: is it an equivalence relation, a partial order, a preorder, . . . ?)

(3.5) Explain in ordinary language what Sxy says about the relationship between x and y, in terms of their R-successor sets.

Optional Challenge

(C.1) In Part 1, we saw that if $Rxy \leftrightarrow Px \wedge Qy$, then R satisfies the Rectangle Law. Prove the *converse*: if a binary relation R satisfies

$$\forall x \forall x' \forall y \forall y' \big((Rxy \wedge Rx'y') \to (Rxy' \wedge Rx'y)\big),$$

then there exist monadic predicates P and Q such that

$$Rxy \leftrightarrow Px \wedge Qy$$

holds in the structure.

(Hint: let Px say "row x of R is nonempty", and let Qy say "column y of R is nonempty".)