Worksheet: Relational Properties and Spans

Background

Let S and R be binary relation symbols. We say that x *and* y *are spanned by* S if there exists a z such that

$$Szx \ \wedge \ Szy.$$

We consider two different ways of relating R and S:

(A) One-way span axiom:

$$\forall x \forall y \big(\exists z (Szx \wedge Syx) \to Rxy \big).$$

This says: if x and y share a common S-predecessor, then Rxy holds.

(B) Definitional equivalence:

$$\forall x \forall y \big(Rxy \leftrightarrow \exists z (Szx \wedge Syx) \big).$$

Here R is *exactly* the span of S.

We investigate which relational properties transfer from S to R under each assumption.

Part I: Countermodels under the One-Way Span Axiom

1. Nothing interesting follows from (A)

Task: Give a structure $\mathcal{M}$ with domain D and interpretations of S and R such that:

1. $\mathcal{M} \models \forall x \forall y (\exists z (Szx \wedge Syx) \to Rxy)$, but
2. R fails to be reflexive, symmetric, and transitive.

Hint: Let S be empty, and let R be *anything at all*. Explain why the implication in (A) is automatically satisfied.

2. A more interesting countermodel

Now give a structure where S *does* have nontrivial spans (i.e. some pairs x, y share an S-predecessor), but R still fails to have any nice property you choose (reflexivity, symmetry, or transitivity).

Write down explicitly:

- domain D,
- extension of S,
- extension of R,
- verification that (A) holds,
- verification that the chosen property of R fails.

Part II: Property Transfer under Definitional Equivalence

Now assume the stronger connection (B):

$$Rxy \quad \text{iff} \quad \exists z(Szx \wedge Syx).$$

3. Symmetry of R

Show that under (B),

$$\forall x \forall y (Rxy \rightarrow Ryx)$$

is valid in all structures.

Task: Prove the sequent

$$\forall x \forall y \big(Rxy \leftrightarrow \exists z(Szx \wedge Syx)\big) \quad \vdash \quad \forall x \forall y (Rxy \rightarrow Ryx)$$

using the HLW/Lemmon natural deduction rules.

4. Failure of Transitivity

Show that even under (B), R need *not* be transitive. Construct a countermodel.

Task: Provide a structure $\mathcal{M}$ such that:

- $\mathcal{M} \models$ (B), i.e. R is *exactly* the S-span;
- $\mathcal{M} \not\models \forall x \forall y \forall z \big((Rxy \wedge Ryz) \rightarrow Rxz\big)$.

Hint: Make three points a, b, c which pairwise share different S-predecessors, but no single predecessor is shared by all three.

Part III: Reflexivity Transfer

5. When does R become reflexive?

We want Rxx to hold for every x. Under (B), this means:

$$\forall x\, Rxx \quad \text{iff} \quad \forall x\, \exists z(Szx \wedge Sxx).$$

Task A: Give conditions on S that ensure $\forall x Rxx$ holds. (Hint: consider *left-seriality* $\forall x \exists z\, Szx$ and *reflexivity* $\forall x\, Sxx$.)

Task B: Prove the following sequent in Lemmon/HLW style:

$$\begin{aligned}&\forall x \forall y \big(Rxy \leftrightarrow \exists z(Szx \wedge Syx)\big),\\ &\forall x Sxx,\\ &\forall x \exists z Szx \quad \vdash \quad \forall x Rxx.\end{aligned}$$

You may assume standard relational equivalences and use EI/EG and UG in the HLW system.

Part IV: Extra Exploration (Optional)

6. Defining spans the other way around

Instead of taking S-predecessors, suppose we define R by common S-*successors*:

$$Rxy \quad \text{iff} \quad \exists z(Sxz \wedge Syz).$$

Tasks:

1. Show that R is automatically symmetric, regardless of what S is.
2. Investigate: under what conditions on S will R be reflexive? transitive?
3. Compare your answers with Parts II–III.